

Solutions to Maths workbook – 1 | Trigonometric Identities and Equations

Level - 3

Daily Tutorial Sheet - 14

162. Writing $x_n = \tan a_n$ for $0^\circ < a_n < 90^\circ$, by the half-angle formula

$$\text{We have } x_{n+1} = \tan a_n + \sqrt{1 + \tan^2 a_n} = \tan a_n + \sec a_n = \frac{1 + \sin a_n}{\cos a_n} = \tan \left(\frac{90^\circ + a_n}{2} \right).$$

Because $a_1 = 60^\circ$, we have $a_2 = 75^\circ$, $a_3 = 82.5^\circ$, and in general $a_n = 90^\circ - \frac{30^\circ}{2^{n-1}}$.

$$\text{Thus } x_n = \tan \left(90^\circ - \frac{30^\circ}{2^{n-1}} \right) = \cot \left(\frac{30^\circ}{2^{n-1}} \right) = \cot \theta_n, \text{ where } \theta_n = \frac{30^\circ}{2^{n-1}}.$$

$$\text{A similar calculation shows that } y_n = \tan 2\theta_n = \frac{2 \tan \theta_n}{1 - \tan^2 \theta_n} \text{ implying that } x_n y_n = \frac{2}{1 - \tan^2 \theta_n}.$$

Because $\theta_n < 45^\circ$, we have $0 < \tan^2 \theta_n < 1$ and $x_n y_n > 2$. For $n > 1$, we have $\theta_n < 30^\circ$, implying that $\tan^2 \theta_n < \frac{1}{3}$ and $x_n y_n < 3$.

163. Dividing both sides of the inequality by 4 yields

$$\left| \frac{a}{2} \sqrt{1 - \left(\frac{b}{2} \right)^2} - \frac{b}{2} \sqrt{1 - \left(\frac{a}{2} \right)^2} \right| \leq \frac{1}{2}.$$

We substitute $\frac{a}{2} = \sin x$ and $\frac{b}{2} = \sin y$. The last inequality reduces to

$$|\sin(x - y)| = |\sin x \cos y - \sin y \cos x| \leq \sin \frac{\pi}{6}.$$

We want to find t_1 and t_2 such that $\sin t_1 = \frac{\sqrt{2} - \sqrt{6}}{4}$ and $\sin t_2 = \frac{\sqrt{2} + \sqrt{6}}{4}$.

By the double-angle formulas, we conclude that $\cos 2t_1 = 1 - 2 \sin^2 t_1 = 1 - \frac{8 - 4\sqrt{3}}{8} = \frac{\sqrt{3}}{2} = \cos \left(\pm \frac{\pi}{6} \right)$ and .

Because $y = \sin x$ is a one-to-one and onto map between the intervals $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $[-1, 1]$, it follows

that $t_1 = -\frac{\pi}{12}$ and $t_2 = \frac{5\pi}{12}$.

We divide the interval $\left[-\frac{\pi}{12}, \frac{5\pi}{12} \right]$ into three disjoint intervals of length $\frac{\pi}{6}$: $I_1 = \left[-\frac{\pi}{12}, \frac{\pi}{12} \right]$, $I_2 = \left[\frac{\pi}{12}, \frac{\pi}{4} \right]$,

and $I_3 = \left[\frac{\pi}{4}, \frac{5\pi}{12} \right]$. The function $y = 2 \sin x$ takes the intervals I_1, I_2, I_3 injectively and surjectively to the

intervals $I'_1 = \left[\frac{\sqrt{2} - \sqrt{6}}{2}, 2 \sin \frac{\pi}{12} \right]$, $I'_2 = \left[2 \sin \frac{\pi}{12}, \sqrt{2} \right]$, $I'_3 = \left[\sqrt{2}, \frac{\sqrt{2} + \sqrt{6}}{2} \right]$, respectively.

By the pigeonhole principle, one of the intervals I'_1, I'_2 or I'_3 contains two of the four given numbers, say a and b . It follows that one of the intervals I_1, I_2 or I_3 contains x and y such that $a = 2 \sin x$ and $b = 2 \sin y$.

164. We can write: $(b \tan \gamma - c \tan \beta)^2 + (c \tan \alpha - a \tan \gamma)^2 + (a \tan \beta - b \tan \alpha)^2$
 $= (a^2 + b^2 + c^2)(\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma) - (a \tan \alpha + b \tan \beta + c \tan \gamma)^2$

The minimum value of the LHS being zero, that of $(a^2 + b^2 + c^2)(\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma) - k^2 \geq 0$

$$\Rightarrow \tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma \geq \frac{k^2}{a^2 + b^2 + c^2}$$

Hence, minimum value of $\tan^2 \alpha + \tan^2 \beta + \tan^2 \gamma$ is $\left(\frac{k^2}{a^2 + b^2 + c^2} \right)$.

165. Given $\tan \frac{1}{4}(\beta + \gamma - \alpha) \tan \frac{1}{4}(\gamma + \alpha - \beta) \tan \frac{1}{4}(\alpha + \beta - \gamma) = 1$,

Where $\alpha + \beta + \gamma = \pi$

$$\Rightarrow \tan \left(\frac{\pi - 2\alpha}{4} \right) \tan \left(\frac{\pi - 2\beta}{4} \right) \tan \left(\frac{\pi - 2\gamma}{4} \right) = 1$$

$$\Rightarrow \left(1 - \tan \frac{\alpha}{2} \right) \left(1 - \tan \frac{\beta}{2} \right) \left(1 - \tan \frac{\gamma}{2} \right) = \left(1 + \tan \frac{\alpha}{2} \right) \left(1 + \tan \frac{\beta}{2} \right) \left(1 + \tan \frac{\gamma}{2} \right)$$

$$\Rightarrow \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = -\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \quad \dots\dots(i)$$

Also, $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1 \quad \dots\dots(ii)$

On squaring equation (i) and using equation (ii): $\left\{ \because \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\pi}{2} \right\}$

$$\tan^2 \frac{\alpha}{2} + \tan^2 \frac{\beta}{2} + \tan^2 \frac{\gamma}{2} + 2 = \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \quad \dots\dots(iii)$$

We have to prove: $1 + \cos \alpha + \cos \beta + \cos \gamma = 0$

$$\Rightarrow 1 + \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} + \frac{1 - \tan^2 \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}} + \frac{1 - \tan^2 \frac{\gamma}{2}}{1 + \tan^2 \frac{\gamma}{2}} = 0 \Rightarrow \frac{2}{1 + \tan^2 \frac{\alpha}{2}} + \frac{2 \left(1 - \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \right)}{\left(1 + \tan^2 \frac{\beta}{2} \right) \left(1 + \tan^2 \frac{\gamma}{2} \right)} = 0$$

$$\Rightarrow \left(1 + \tan^2 \frac{\alpha}{2} \right) \left(1 - \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \right) + \left(1 + \tan^2 \frac{\beta}{2} \right) \left(1 + \tan^2 \frac{\gamma}{2} \right) = 0$$

$$\Rightarrow \left(1 + \tan^2 \frac{\alpha}{2} - \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} - \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \right) + \left(1 + \tan^2 \frac{\beta}{2} + \tan^2 \frac{\gamma}{2} + \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \right) = 0$$

$$\Rightarrow 2 + \tan^2 \frac{\alpha}{2} + \tan^2 \frac{\beta}{2} + \tan^2 \frac{\gamma}{2} = \tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2} \tan^2 \frac{\gamma}{2} \quad \dots\dots(iv)$$

From equation (iii) and (iv)

$$1 + \cos \alpha + \cos \beta + \cos \gamma = 0$$

166. $\{\sin(\alpha - \beta) + \cos(\alpha + 2\beta) \cdot \sin \beta\}^2 = 4 \cos \alpha \cdot \sin \beta \cdot \sin(\alpha + \beta)$

$$\Rightarrow \{\sin \alpha \cos \beta - \sin \beta \cos \alpha + (\cos \alpha \cos 2\beta - \sin \alpha \sin 2\beta) \sin \beta\}^2 = 4 \cos \alpha \sin \beta \sin(\alpha + \beta)$$

$$\Rightarrow \{\tan \alpha - \tan \beta + \cos 2\beta \cdot \tan \beta - \sin 2\beta \cdot \tan \alpha \tan \beta\}^2 = 4 \tan \beta (\tan \alpha + \tan \beta) \quad [\because \text{Dividing by } \cos^2 \alpha \cdot \cos^2 \beta]$$

$$\Rightarrow \{\tan \alpha \cdot \cos 2\beta - \tan \beta + \cos 2\beta \cdot \tan \beta\}^2 = 4 \tan \beta (\tan \alpha + \tan \beta)$$

$$\Rightarrow \{(\tan \alpha + \tan \beta) \cdot \cos 2\beta - \tan \beta\}^2 = 4 \tan \beta (\tan \alpha + \tan \beta) \quad \dots\dots\dots(i)$$

If $(\tan \alpha + \tan \beta) = \frac{\tan \beta}{x} \quad \dots\dots\dots(ii)$

Equation (i) becomes; $\left\{ \frac{\tan \beta}{x} \cdot \cos 2\beta - \tan \beta \right\}^2 = 4 \tan \beta \cdot \frac{\tan \beta}{x}$

$$\Rightarrow (\cos 2\beta - x)^2 = 4x \Rightarrow \cos^2 2\beta + x^2 - 2x \cos 2\beta = 4x$$

$$\Rightarrow x^2 - 2x(\cos 2\beta + 2) + \cos^2 2\beta = 0 \Rightarrow x = (\cos 2\beta + 2) \pm \sqrt{1 + \cos 2\beta}$$

$$\Rightarrow x - \cos 2\beta + 2 \pm 2\sqrt{2 \cos^2 \beta} \Rightarrow x = 2 \cos^2 \beta - 1 + 2 \pm 2\sqrt{2} \cos \beta = (\sqrt{2} \cos \beta \pm 1)^2$$

$$\Rightarrow \tan \alpha + \tan \beta = \frac{\tan \beta}{x} = \frac{\tan \beta}{(\sqrt{2} \cos \beta - 1)^2} \quad [\text{since, } x < 1]$$

167. $x = \tan(n\theta + \alpha) - \tan(n\theta + \beta) = \frac{\sin(n\theta + \alpha)}{\cos(n\theta + \alpha)} - \frac{\sin(n\theta + \beta)}{\cos(n\theta + \beta)} = \frac{\sin(n\theta + \alpha - n\theta - \beta)}{\cos(n\theta + \alpha)\cos(n\theta + \beta)} = \frac{2 \sin(\alpha - \beta)}{\cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta)}$

$$\Rightarrow \cos(2n\theta + \alpha + \beta) + \cos(\alpha - \beta) = \frac{2 \sin(\alpha - \beta)}{x} \quad \dots\dots\dots(i)$$

Again $y = \cot(n\theta + \alpha) - \cot(n\theta + \beta) = \frac{\cos(n\theta + \alpha)}{\sin(n\theta + \alpha)} - \frac{\cos(n\theta + \beta)}{\sin(n\theta + \beta)} = \frac{\sin(n\theta + \beta - n\theta - \alpha)}{\sin(n\theta + \alpha)\sin(n\theta + \beta)}$

$$\Rightarrow y = \frac{2 \sin(\beta - \alpha)}{\cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta)} \Rightarrow \cos(\alpha - \beta) - \cos(2n\theta + \alpha + \beta) = \frac{2 \sin(\beta - \alpha)}{y} \quad \dots\dots\dots(ii)$$

On adding equations (i) and (ii), we get: $2 \cos(\alpha - \beta) = \frac{2 \sin(\alpha - \beta)}{x} + \frac{2 \sin(\beta - \alpha)}{y} \Rightarrow \cot(\alpha - \beta) = \frac{1}{x} - \frac{1}{y}$